Mathematical Induction

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Notation

Statements

We use P_n to denote a statement. The statement should depend on n to avoid wasting the valuable subscript. For example, if we define:

$$P_n := \text{There are } n \text{ lights.}$$

Then, we would have:

$$\begin{aligned} P_2 &= \text{There are 2 lights.} \\ P_7 &= \text{There are 7 lights.} \\ P_{126} &= \text{There are 126 lights.} \end{aligned}$$

For another example, if we define:

$$P_n := n^2 + n \text{ is even.}$$

Then, we would have:

$$\begin{aligned} P_1 &= 1^2 + 1 \text{ is even.} \\ P_8 &= 8^2 + 8 \text{ is even.} \\ P_{911} &= 911^2 + 911 \text{ is even.} \end{aligned}$$

Inductive Claims

We use I_n to denote an inductive claim. An inductive claim I_n says that if P_n is true, then P_{n+1} must be true. In other words, we define:

$$I_n := (P_n \implies P_{n+1}).$$

For example, suppose we define:

$$P_n := n^3 \text{ is prime.}$$

Then, we would have:

$$\begin{aligned} I_2 &= \text{If } 2^3 \text{ is prime, then } 3^3 \text{ is prime.} \\ I_{12} &= \text{If } 12^3 \text{ is prime, then } 13^3 \text{ is prime.} \\ I_{42} &= \text{If } 42^3 \text{ is prime, then } 42^3 \text{ is prime.} \end{aligned}$$

The Principle of Mathematical Induction

The Principle of Mathematical Induction (PMI) says that if P_1 is true, and I_n is true for all natural numbers n, then P_n is true for all natural numbers n. In other words, we have that:

$$[P_1 \text{ and } (I_n \text{ for all } n \in \mathbb{N})] \implies (P_n \text{ for all } n \in \mathbb{N})$$

Or, more succinctly,

$$P_1 \wedge (I_n \forall n \in \mathbb{N}) \implies (P_n \forall n \in \mathbb{N})$$

To see why this is true, let us consider:

$$P_n := n > 0$$

- The PMI requires that we have P_1. In other words, we must assume that $1 > 0$.
- We also assume that we have proved I_1. In other words, we have that if $1 > 0$, then $2 > 0$.
- Clearly, it follows that $2 > 0$. In other words, we have proved P_2 from P_1 and I_1.

We can repeat this reasoning for P_2 and I_2 to obtain P_3, and so on.

The statement P_1 is called the *base case*, and proving the inductive claim I_n for all $n \in \mathbb{N}$ is called the *inductive step*.

Examples

Sums of Odd Numbers

Note that:

$$\begin{aligned} 1 &= 1 \\ 1+3 &= 4 \\ 1+3+5 &= 9 \\ 1+3+5+7 &= 16 \\ 1+3+5+7+11 &= 25 \\ &\vdots \end{aligned}$$

It appears that the sum of the first n odd numbers is equal to n^2. We want to prove this.
Theorem 1. *The sum of the first n odd numbers is equal to n^2 for all $n \in \mathbb{N}$.*

Proof. ??????????????????????????? ∎

We will prove this by induction; however, we must first formalize the problem. We do this by defining:

$$P_n := \text{The sum of the first } n \text{ odd numbers is equal to } n^2.$$

The problem has now been transformed into proving P_n for all $n \in \mathbb{N}$.
Theorem 2. *P_n is true for all $n \in \mathbb{N}$.*

Proof. ??????????????????????????? ∎

This presentation is on induction, so we will clearly use induction to prove P_n. The PMI says that we only need to prove P_1 and I_n.
Lemma 3. *P_1 is true.*

Proof. Substituting $n = 1$ into the definition of P_1 yields:

$$P_1 = \text{The sum of the first 1 odd numbers is equal to } 1^2.$$

In other words,

$$P_1 = \text{The first odd number is 1.}$$

This is true by the "duh of course" principle. ∎

We have proved that P_1 is true. We now need to prove that I_n is true for any n.
Lemma 4. *I_n is true for all $n \in \mathbb{N}$.*

Proof. ??????????????????????????? ∎

It might help to substitute the definitions of I_n and P_n:
Lemma 5. *If the sum of the first n odd numbers is n^2, then the sum of the first $n+1$ odd numbers is $(n+1)^2$.*

Proof. ??????????????????????????? ∎

It might also help to actually know what the first n odd numbers are. The first odd number is $1 \times 2 - 1 = 1$. The second odd number is $2 \times 2 - 1 = 3$. The third odd number is $3 \times 2 - 1 = 5$. It appears that the n^{th} odd number is $2n - 1$. We can prove this by induction but I'm too lazy. It follows that the sum of the first n odd numbers is the sum of the numbers $2k - 1$, where k goes from 1 to n.
Lemma 6. *If the sum of the numbers $2k - 1$, where k goes from 1 to n, is n^2, then the sum of the numbers $2k - 1$, where k goes from 1 to $n + 1$, is $(n+1)^2$.*

Proof. ??????????????????????????? ∎

This is too wordy, so let's use more symbols.
Lemma 7. *If*

$$\sum_{k=1}^{n}(2k-1) = n^2,$$

then

$$\sum_{k=1}^{n+1}(2k-1) = (n+1)^2$$

Proof. ??????????????????????????? ∎

We're getting somewhere (hopefully). Actually, why don't we try to prove it now (maybe?).
Lemma 8. *If*

$$\sum_{k=1}^{n}(2k-1) = n^2,$$

then

$$\sum_{k=1}^{n+1}(2k-1) = (n+1)^2$$

Proof. We know that:

$$\sum_{k=1}^{n}(2k-1) = n^2.$$

We want to end up with $\sum_{k=1}^{n+1}(2k-1)$ on the left. This is the same sum but with a $2(n+1) - 1$ added. Let's add $2(n+1) - 1$ to both sides.

$$\begin{aligned} \sum_{k=1}^{n}(2k-1) &= n^2 \\ \sum_{k=1}^{n}(2k-1) + 2(n+1) - 1 &= n^2 + 2(n+1) - 1 \\ \sum_{k=1}^{n+1}(2k-1) &= n^2 + 2n + 2 - 1 \\ \sum_{k=1}^{n+1}(2k-1) &= (n+1)^2 \end{aligned}$$

We got what we wanted assuming only what we were given, so I'd call that a proof. ∎

We've managed to prove both P_1 and I_n. PMI tells us that this proves P_n as well, so we're done.

Here's the full proof, for reference.
Theorem 9. *The sum of the first n odd numbers is equal to n^2 for all $n \in \mathbb{N}$.*

Proof. We will use the Principle of Mathematical Induction.

Base Case The sum of the first 1 odd numbers equals 1^2. The claim holds.

Inductive Step We will assume the claim is true for some $n \in \mathbb{N}$. Note that:

$$\begin{aligned} \sum_{k=1}^{n}(2k-1) &= n^2 \\ \sum_{k=1}^{n}(2k-1) + 2(n+1) - 1 &= n^2 + 2(n+1) - 1 \\ \sum_{k=1}^{n+1}(2k-1) &= (n+1)^2 \end{aligned}$$

The claim holds for $n + 1$ as well. By PMI, the claim holds for all $n \in \mathbb{N}$.

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Exponentials versus Factorial Growth

Look at the following table:

n	2^n	$n!$
1	2	1
2	4	2
3	8	6
4	16	24
5	32	120
6	64	720
7	128	5040
8	256	40320
9	512	362880
10	1024	3628800
11	2048	39916800

It looks like $n!$ grows faster than 2^n. In particular, it appears that $n! > 2^n$ for all $n \geq 4$. We can prove this by a modified version of PMI; we will define:

$$P_n := n! > 2^n$$

We only need to prove P_4 and I_n for all $n \geq 4$. This will show that P_n is true for $n \geq 4$.
Theorem 10. *$n! > 2^n$ for all $n \geq 4$.*

Proof. We will use the Principle of Mathematical Induction.

Base Case We have that $4! > 2^4$. The claim holds.

Inductive Step We will assume the claim is true for some $n \geq 4$. Note that:

$$\begin{aligned} n! &> 2^n \\ n!\,(n+1) &> (n+1)2^n \\ (n+1)! &> (n+1)2^n \\ (n+1)! &> 2 \times 2^n && (\text{since } n + 1 > 2) \\ (n+1)! &> 2^{n+1} \end{aligned}$$

The claim holds for $n + 1$ as well. By PMI, the claim holds for all $n \geq 4$.

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An alternative proof of the theorem is as follows:
Theorem 11. *$n! > 2^n$ for all $n \geq 4$.*

Proof. We will use the Principle of Mathematical Induction.

Base Case We have that $4! > 2^4$. The claim holds.

Inductive Step We will assume the claim is true for some $n \geq 4$. Note that:

$$(n+1)! = n!\,(n+1) > n! \times 2 > 2^n \times 2 = 2^{n+1}$$

The claim holds for $n + 1$ as well. By PMI, the claim holds for all $n \geq 4$.

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All Numbers in a List are the Same

We will prove that given any list of n numbers, all the elements of the list are the same.
Theorem 12. *In a list of n numbers, all the numbers are the same.*

Proof. We will use the Principle of Mathematical Induction.

Base Case In a list with one element, all the elements in the list are equal (since there's only one). The claim holds.

Inductive Step We will assume the claim is true for some $n \in \mathbb{N}$. Consider any list of length $n + 1$.

$$\overbrace{5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5}^{n \text{ equal elements}} \quad 5$$
$$5 \quad \underbrace{5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5}_{n \text{ equal elements}}$$

The first n elements are all the same, by the induction hypothesis. The last n elements are also all the same. We therefore must have all $n + 1$ elements be the same; the claim holds for $n + 1$ as well. By PMI, the claim holds for all $n \in \mathbb{N}$.

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Divisibility Examples

We will show that $4^n + 2$ is divisible by 3 for all $n \geq 0$. To formalize the problem, we will define:

$$P_n := 4^n + 2 \text{ is divisible by 3.}$$

What "x is divisible by 3" means is that x is equal to three times an integer, so we can rewrite P_n as follows:

$$P_n = (4^n + 2 = 3k \text{ for some } k \in \mathbb{Z}).$$

Theorem 13. *$4^n + 2$ is divisible by 3 for all $n \geq 0$.*

Proof. We will use the Principle of Mathematical Induction.

Base Case $4^0 + 2 = 3$ is divisible by 3. The claim holds.

Inductive Step We will assume the claim is true for some $n \in \mathbb{N}$. It follows that $4^n + 2 = 3k$ for some $k \in \mathbb{Z}$. Note that:

$$\begin{aligned} 4^n + 2 &= 3k \\ 4 \times (4^n + 2) &= 4 \times 3k \\ 4 \times 4^n + 4 \times 2 &= 12k \\ 4^{n+1} + 8 &= 12k \\ 4^{n+1} + 2 &= 12k - 6 \\ 4^{n+1} + 2 &= 3(4k - 2) \end{aligned}$$

We obtain that $4^{n+1} + 2$ is three times an integer, so it must be divisible by 3. It follows that the claim holds for $n + 1$ as well. By PMI, the claim holds for all $n \geq 0$.

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We will now present an incomplete proof for a similar fact. Should your algebra skills be up to par, you should be able to fill in any missing steps (in red).
Theorem 14. *$n^3 + 2n$ is divisible by 3 for all $n \in \mathbb{N}$.*

Proof. We will use the Principle of Mathematical Induction.

Base Case Missing Step

Inductive Step We will assume the claim is true for some $n \in \mathbb{N}$. It follows that $n^3 + 2n = 3k$ for some $k \in \mathbb{Z}$. Note that:

$$n^3 + 2n = 3k$$

Missing Step

$$\begin{aligned} (n^3 + 3n^2 + 3n + 1) + (2n + 2) &= 3 \times ??? \\ (n+1)^3 + 2(n+1) &= 3 \times ??? \end{aligned}$$

We obtain that $(n+1)^3 + 2(n+1)$ is three times an integer, so it must be divisible by 3. It follows that the claim holds for $n + 1$ as well. By PMI, the claim holds for all $n \in \mathbb{N}$.

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The next example has a small caveat; instead of claiming that P_n is true for some n onwards, we prove the statement for all odd n. We use modified PMI with a base case of $n = 1$ and inductive claims of the form $P_n \implies P_{n+2}$.
Theorem 15. *If n is odd, then $(n+2)^2 - n^2$ is a multiple of 8.*

Proof. We will use the Principle of Mathematical Induction.

Base Case $(1+2)^2 - 1^2 = 8$ is divisible by 8, so the claim holds.

Inductive Step We will assume the claim is true for some $n \in \mathbb{N}$. It follows that $(n+2)^2 - n^2 = 8k$ for some $k \in \mathbb{Z}$. Note that:

$$\begin{aligned} (n+2)^2 - n^2 &= 8k \\ (n+2)^2 - n^2 + 2 \times 2(n+2) + 2^2 - 4n - 4 &= \\ 8k + 2 \times 2(n+2) + 2^2 - 4n - 4 & \\ [(n+2)^2 + 2 \times 2(n+2) + 2^2] - (n^2 + 4n + 4) &= 8k + 8 \\ [(n+2) + 2]^2 - (n+2)^2 &= 8(k+1) \end{aligned}$$

We obtain that $[(n+2)+2]^2 - (n+2)^2$ is 8 times an integer, so it must be divisible by 8. It follows that the claim holds for $n + 2$ as well. By PMI, the claim holds for all odd n.

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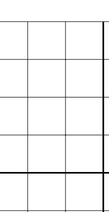
Checkerboard Tiling

For any $n \in \mathbb{N}$, consider a $2^n \times 2^n$ checkerboard with one of the squares randomly removed. We will prove that it is possible to exactly cover the board with the shape below.



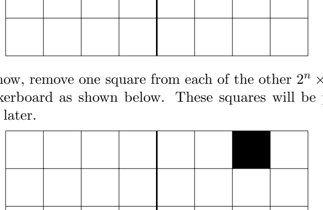
Theorem 16. *For any $n \in \mathbb{N}$, it is possible to tile a $2^n \times 2^n$ checkerboard that has any square removed using only the above piece.*

Proof. We will use the Principle of Mathematical Induction.

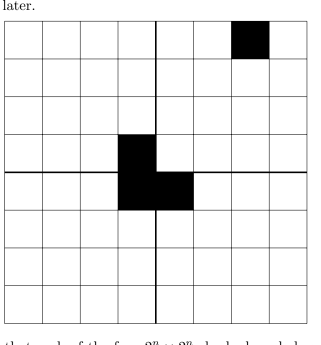
Base Case For $n = 1$, the checkerboard is 2×2. Clearly, if any square is removed, the remaining board is exactly the above shape, and so can be covered by one piece. The claim holds.

Inductive Step We will assume the claim is true for some $n \in \mathbb{N}$.

Consider a checkerboard of size $2^{n+1} \times 2^{n+1}$. Divide the checkerboard into four $2^n \times 2^n$ checkerboards, as shown below. For now, assume that the missing square is in the upper right checkerboard; the proof is similar for any other case.



For now, remove one square from each of the other $2^n \times 2^n$ checkerboard as shown below. These squares will be put back later.



Note that each of the four $2^n \times 2^n$ checkerboards has one square missing. By the induction hypothesis, each of them can be covered exactly with the pieces described. The three squares we removed can be covered with a single additional piece. This yields a covering of the whole board with the L-shaped pieces.

It follows that the claim holds for $n + 1$ as well. By PMI, the claim holds for all odd n.

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Binomial Theorem

Before we prove the binomial theorem using induction, we will first prove a small lemma:
Lemma 17. *For all $n, k \in \mathbb{N}$ such that $0 \leq k \leq n$, we have that:*

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Proof. Consider the row of n balls below. One of them is black, and the rest are white. Clearly, there are $n - 1$ white balls.

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Note the following:

- There are $\binom{n}{k}$ ways to choose k of the n balls above.
- There are $\binom{n-1}{k}$ ways to choose k of only the white balls.
- There are $\binom{n-1}{k-1}$ ways to choose the black ball and $k - 1$ of the white balls.

The only way we can choose k balls is to either choose only white ones, or to include the black one. Therefore, it follows that:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

∎

We can now prove the binomial theorem inductively.
Theorem 18. *For all $n \in \mathbb{N}$,*

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$$

Proof. We will use the Principle of Mathematical Induction.

Base Case For $n = 1$, we have that:

$$\begin{aligned} \sum_{k=0}^{1}\binom{1}{k}a^k b^{1-k} &= \binom{1}{0}a^0 b^{1-k} + \binom{1}{1}a^1 b^{1-1} \\ &= b + a \\ &= (a+b)^1 \end{aligned}$$

The claim holds.

Inductive Step We will assume the claim is true for some $n \in \mathbb{N}$.

Note that:

$$\begin{aligned} (a+b)^n &= \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} \\ (a+b) \times (a+b)^n &= (a+b)\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} \\ (a+b)^{n+1} &= a\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} + b\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} \\ (a+b)^{n+1} &= \sum_{k=0}^{n}\binom{n}{k}a^{k+1} b^{n-k} \\ &\quad + \sum_{k=0}^{n}\binom{n}{k}a^k b^{(n+1)-k} \\ (a+b)^{n+1} &= \sum_{k=1}^{n+1}\binom{n}{k-1}a^k b^{n-(k-1)} \\ &\quad + \sum_{k=0}^{n}\binom{n}{k}a^k b^{(n+1)-k} \\ (a+b)^{n+1} &= \binom{n}{n}a^{n+1}b^0 + \sum_{k=1}^{n}\binom{n}{k-1}a^k b^{n-(k-1)} \\ &\quad + \binom{n}{0}a^0 b^{n+1} + \sum_{k=1}^{n}\binom{n}{k}a^k b^{(n+1)-k} \\ (a+b)^{n+1} &= a^{n+1} + b^{n+1} \\ &\quad + \sum_{k=1}^{n}\left[\binom{n}{k-1}a^k b^{n-(k-1)} + \binom{n}{k}a^k b^{(n+1)-k}\right] \\ (a+b)^{n+1} &= a^{n+1} + b^{n+1} \\ &\quad + \sum_{k=1}^{n}\left[\binom{n}{k-1} + \binom{n}{k}\right]a^k b^{(n+1)-k} \\ (a+b)^{n+1} &= a^{n+1} + b^{n+1} \\ &\quad + \sum_{k=1}^{n}\binom{n+1}{k}a^k b^{(n+1)-k} \\ (a+b)^{n+1} &= \sum_{k=0}^{n+1}\binom{n+1}{k}a^k b^{(n+1)-k} \end{aligned}$$

It follows that the claim holds for $n + 1$ as well. By PMI, the claim holds for all $n \in \mathbb{N}$.

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Exercise

Theorem 19. *If $r \neq 1$, then for all $n \in \mathbb{N}$:*

$$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}.$$

Proof. We will use the Principle of Mathematical Induction.

Base Case Missing Step

Inductive Step We will assume the claim is true for some $n \in \mathbb{N}$. Note that:

$$\begin{aligned} \sum_{k=0}^{n-1} r^k &= \frac{1-r^n}{1-r} \\ \sum_{k=0}^{n-1} r^k + r^n &= \frac{1-r^n}{1-r} + r^n \end{aligned}$$

Missing Step

$$\sum_{k=0}^{(n+1)-1} r^k = \frac{1-r^{n+1}}{1-r}$$

It follows that the claim holds for $n + 1$ as well. By PMI, the claim holds for all $n \in \mathbb{N}$.

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